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ON THERMOCAPILLARY CONVECTION OF A LIQUID IN A FLOATING ZONE

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A new approach to the study of the thermocapillary convection of a liquid in a floating zone is proposed.

The study of the thermocapillary convection of a liquid in a floating zone (a liquid bridge) is an important modern problem of hydromechanics. It is directly related to the process of production of high-quality materials. This area of hydromechanics has been studied rather extensively (see, for example, [1–3] and the bibliography presented there). However, up to now there have not been analytical results that describe the flow of a liquid in a floating zone. This circumstance is due to the fact that the problem is extremely complicated, in particular, because part of the boundary of the region occupied by the liquid is solid, and part of it is free.

In the present paper, we propose an approach that allows one to perform an effective analytical study of the thermocapillary convection of a liquid in a floating zone. This approach uses the engagement phenomenon [1], which is as follows. At the sharp edge of a solid body, the boundary angle (the angle between the free boundary of the liquid and the wetted surface of the solid body) does not have a unique possible value but various values are permissible.

We consider the problem of plane thermocapillary convection of a liquid in a floating zone.

There is a liquid that bounds a gas medium and solid bodies (see Fig. 1). The region Ω occupied by the liquid is an infinitely long cylinder. The generatrices of the cylindrical surface are parallel to the Z axes of a rectangular coordinate system X, Y, Z. The solid bodies have sharp edges, which intersect the plane $Z = 0$ at the points O_1, O_2, O_3, and O_4. Each of these edges coincides with the line of contact of the liquid, the gas, and the solid body. The free boundary of the region Ω consists of two parts: Γ_{f1} and Γ_{f2}. The solid boundary of region Ω consists of two parts: Γ_{s1} and Γ_{s2}. The lines L_{s1} and L_{s2} of intersection of Γ_{s1} and Γ_{s2} with the plane $Z = 0$ are arcs of length $2A\theta^*$ $(0 < \theta^* < \pi/2)$ of a circle of radius A with center at the origin of the coordinates X, Y, Z. The temperature T of the liquid is T_f on Γ_{f1} and Γ_{f2} and T_s on Γ_{s1} and Γ_{s2} (T_f can have different values at different points of Γ_{f1} and Γ_{f2}; T_s is a constant). The surface tension σ of the liquid on the boundary with the gas medium depends on T_f.

For an undisturbed state of the liquid, i.e., for $T_f = T_s$, the region Ω is an infinitely long circular cylinder of radius A, the liquid is at rest, the liquid pressure is constant, and $T = T_s$.

The fact that for the undisturbed state of the liquid the region Ω is a circular cylinder is essential, and this is realized owing to the phenomenon of catching.

At the sharp edge of the solid body, the boundary angle α can have any value that satisfies the condition $\beta \leqslant \alpha \leqslant \beta + \pi - \gamma$, where β is the boundary angle on the smooth surface of the body and γ is the angle between the planes that emerge from the sharp edge of the body and are tangents to the surface of the body. The region occupied by the liquid in the undisturbed state (the circular cylinder) can be produced from another permissible (cylindrical) region occupied by the liquid in the undisturbed state by changing (adding or removing) the amount of the liquid (in each part of finite length of the cylindrical region).

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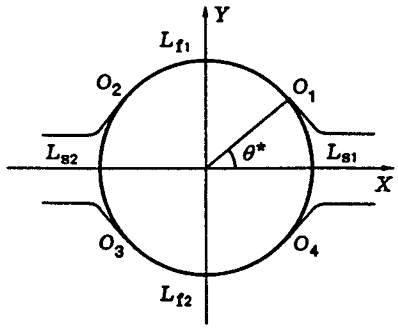

Fig. 1

For $T_f \neq T_s$, the liquid performs stationary motion relative to the coordinates X, Y, Z. The liquid flow is planar (the flow planes are perpendicular to the Z axis). The cross section $Z = 0$ of the region Ω, the temperature, liquid velocity, and pressure in this cross section are symmetric about the X and Y axes.

We assume that $x_1 = X/A$, $x_2 = Y/A$, $x_3 = Z/A$, $r = \sqrt{x_1^2 + x_2^2}$, Γ is the boundary of the region Ω, L is the line of intersection of Γ with the plane $x_3 = 0$, H is the curvature of L, $\eta = A\mathrm{H}$, $\mathbf{e}_n$ is a unit vector normal to Γ (directed from Ω), $\mathbf{e}_t$ is a unit vector tangent to L (directed in the positive direction of tour around L), S is the length of the arc L which issues from the point $(1,0,0)$ (the direction of increase in S coincides with the direction of $\mathbf{e}_t$), $s = S/A$, σ_0 is the value of σ for $T_f = T_s$, $\hat{\sigma}$ is the largest value of $|\sigma - \sigma_0|$, $\sigma = \sigma_0 + \hat{\sigma} f$ $[f = f(s)]$, ν is the kinematic viscosity of the liquid, $\mathrm{Ma} = A\hat{\sigma}/(\rho\nu^2)$ is the Marangoni number, $\lambda = A\sigma_0/(\rho\nu^2)$, $\mathbf{V}$ is the liquid velocity, $\mathbf{v} = A\mathbf{V}/\nu$, ρ is the liquid density, P_g is the gas pressure, P is the liquid pressure, $p = A^2(P - P_g - \sigma_0/A)/(\rho\nu^2)$, $\mathbf{P}$ is the stress tensor of the liquid, $\mathbf{I} = (I_{ij})$ is a unit tensor, $\mathbf{p} = (p_{ij}) = A^2[\mathbf{P} + (P_g + \sigma_0/A)\mathbf{I}]/(\rho\nu^2)$ $(p_{ij} = -pI_{ij} + \partial v_i/\partial x_j + \partial v_j/\partial x_i)$, $\tau = T - T_s$, and χ is the thermal diffusivity of the liquid.

The equations of the lines L_{f1} and L_{f2} of intersection of Γ_{f1} and Γ_{f2} with the plane $x_3 = 0$, the equations of liquid convection (Navier–Stokes continuity, and heat-transfer equations, [4]) and the conditions that should be satisfied on L_{s1}, L_{s2}, L_{f1}, and L_{f2}, have the following form:

$$r = \xi_1; \tag{1}$$

$$r = \xi_2; \tag{2}$$

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \Delta \mathbf{v}; \tag{3}$$

$$\nabla \cdot \mathbf{v} = 0; \tag{4}$$

$$\mathbf{v} \cdot \nabla \tau = \frac{\chi}{\nu} \Delta \tau; \tag{5}$$

$$\mathbf{v} = 0, \quad \tau = 0 \quad \text{on} \quad L_{s1}, \quad L_{s2}; \tag{6}$$

$$\begin{gathered} \mathbf{v} \cdot \mathbf{e}_n = 0, \quad \mathbf{p} \cdot \mathbf{e}_n = [\lambda(1 - \eta) - \mathrm{Ma}\,\eta f]\mathbf{e}_n + \mathrm{Ma}\frac{df}{ds}\mathbf{e}_t, \\ \tau = T_f - T_s \quad \text{on} \quad L_{f1}, \quad L_{f2}. \end{gathered} \tag{7}$$

It is required to determine ξ_1, ξ_2, $\mathbf{v}$, p, and τ.

Let us examine problem (1)–(7) for small Ma numbers compared to unity.

We assume that as $\mathrm{Ma} \to 0$,

$$\begin{gathered} \xi_1 \sim \xi_1^{(0)} + \mathrm{Ma}\,\xi_1^{(1)}, \quad \xi_2 \sim \xi_2^{(0)} + \mathrm{Ma}\,\xi_2^{(1)}, \quad \mathbf{v} \sim \mathbf{v}^{(0)} + \mathrm{Ma}\,\mathbf{v}^{(1)}, \\ p \sim p^{(0)} + \mathrm{Ma}\,p^{(1)}, \qquad \tau \sim \tau^{(0)} + \mathrm{Ma}\,\tau^{(1)}. \end{gathered} \tag{8}$$

In a zero approximation that corresponds to the undisturbed state of the liquid, we have

$$r = \xi_1^{(0)} \qquad (\theta^* < \theta < \pi - \theta^*),$$

which is the equation of the line $L_{f1}^{(0)}$,

$$\xi_1^{(0)} = 1,$$

and

$$r = \xi_2^{(0)} \qquad (\pi + \theta^* < \theta < 2\pi - \theta^*),$$

which is the equation of the line $L_{f2}^{(0)}$,

$$\xi_2^{(0)} = 1$$

[θ is the angle between the vectors $(1,0,0)$ and $(x_1, x_2, 0)$ $(0 \leqslant \theta \leqslant 2\pi)$, and $L_{f1}^{(0)}$ and $L_{f2}^{(0)}$ are the lines of intersection of the free boundary of the region occupied by the liquid in the zero approximation with the plane $x_3 = 0$],

$$\mathbf{v}^{(0)} = 0, \quad p^{(0)} = 0, \quad \tau^{(0)} = 0. \tag{9}$$

Let us define the problem of the first approximation using (1)–(9):

$$r = 1 + \mathrm{Ma}\,\xi_1^{(1)} \qquad (\theta^* < \theta < \pi - \theta^*), \tag{10}$$

which is the equation of the line $L_{f1}^{(1)}$;

$$r = 1 + \mathrm{Ma}\,\xi_2^{(1)} \qquad (\pi + \theta^* < \theta < 2\pi - \theta^*), \tag{11}$$

which is the equation of the line $L_{f2}^{(1)}$ ($L_{f1}^{(1)}$ and $L_{f2}^{(1)}$ are the lines of intersection of the free boundary of the region occupied by the liquid in the first approximation with the plane $x_3 = 0$);

$$-\nabla p^{(1)} + \Delta \mathbf{v}^{(1)} = 0; \tag{12}$$

$$\nabla \cdot \mathbf{v}^{(1)} = 0; \tag{13}$$

$$v_r^{(1)} = 0 \quad \text{on} \quad L_{si}, \ L_{fi}^{(0)} \quad (i = 1, 2); \tag{14}$$

$$v_\theta^{(1)} = 0 \quad \text{on} \quad L_{si} \quad (i = 1, 2); \tag{15}$$

$$-p^{(1)} + 2\frac{\partial v_r^{(1)}}{\partial r} - \lambda\Big(\frac{d^2\xi_i^{(1)}}{d\theta^2} + \xi_i^{(1)}\Big) + f = 0 \ \text{on} \ L_{fi}^{(0)} \ (i = 1, 2); \tag{16}$$

$$\frac{\partial v_\theta^{(1)}}{\partial r} - v_\theta^{(1)} - \frac{df}{d\theta} = 0 \ \text{on} \ L_{fi}^{(0)} \quad (i = 1, 2); \tag{17}$$

$$\Delta \tau^{(1)} = 0; \tag{18}$$

$$\tau^{(1)} = 0 \quad \text{on} \quad L_{si} \qquad (i = 1, 2); \tag{19}$$

$$\tau^{(1)} = \varphi \quad \text{on} \quad L_{fi}^{(0)} \qquad (i = 1, 2). \tag{20}$$

Here $v_r^{(1)}$ and $v_\theta^{(1)}$ are the r and θ components of the vector $\mathbf{v}^{(1)}$, $\varphi = \lim\limits_{\mathrm{Ma}\to 0}(T_f - T_s)/\mathrm{Ma}$. For the region occupied by the liquid in the zero approximation, Eqs. (12), (13) have the following solution that satisfies condition (14):

$$v_r^{(1)} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad v_\theta^{(1)} = -\frac{\partial\psi}{\partial r}; \tag{21}$$

$$p^{(1)} = a_0 + 4\sum_{m=1}^{\infty}(m+1)(a_m \sin m\theta + b_m \cos m\theta)r^m. \tag{22}$$

Here $\psi = (1 - r^2)\sum_{m=1}^{\infty}(a_m \cos m\theta + b_m \sin m\theta)r^m$ and a_0, a_m, and b_m are constants. Using (15) and (21), we obtain

$$a_m = 0 \qquad (m = 1, 2, \ldots); \tag{23}$$

$$b_{2n-1} = 0 \qquad (n = 1, 2, \ldots); \tag{24}$$

$$b_{2n} = \frac{2}{\pi}\int_{\theta^*}^{\pi/2} u \sin 2n\theta d\theta \qquad (n = 1, 2, \ldots), \tag{25}$$

where $u = v_\theta^{(1)}\big|_{r=1,\, \theta^* \leqslant \theta \leqslant \pi/2}$. We assume that $T_f \to T_s$ as $s \to \theta^* + 0$. Accordingly, we have

$$f = 0 \quad \text{for } \theta = \theta^*, \ \theta = \pi - \theta^*, \ \theta = \pi + \theta^*, \ \theta = 2\pi - \theta^*. \tag{26}$$

From (17), (21), and (23)–(26) it follows that

$$f = 4\sum_{n=1}^{\infty} b_{2n}(\cos 2n\theta^* - \cos 2n\theta). \tag{27}$$

Equalities (25) and (27) define the relationship between the quantities f and u. According to (17), (21), and (23)–(25), we have

$$b_{2k} = \frac{1}{2k(\pi - 2\theta^*) + \sin 4k\theta^*}$$

$$\times\Big\{\int_{\theta^*}^{\pi/2}\frac{df}{d\theta}\sin 2k\theta d\theta + 2\sum_{\substack{n\neq k\\ n=1}}^{\infty}\frac{n}{n^2-k^2}[(n+k)\sin 2(n-k)\theta^* - (n-k)\sin 2(n+k)\theta^*]b_{2n}\Big\} \quad (k = 1, 2, \ldots).$$

Using (16) and (21)–(25) and taking into account that the lines $L_{f1}^{(1)}$ and $L_{f2}^{(1)}$ border the lines L_{s1} and L_{s2} at the points O_1, O_2, O_4, and O_3, we obtain

$$\xi_1^{(1)} = \mu\Big(1 - \frac{\sin\theta}{\sin\theta^*}\Big), \qquad \xi_2^{(1)} = \mu\Big(1 + \frac{\sin\theta}{\sin\theta^*}\Big), \tag{28}$$

where

$$\mu = \frac{1}{\lambda}\Big(-a_0 + 4\sum_{n=1}^{\infty} b_{2n}\cos 2n\theta^*\Big).$$

Let, in the first approximation, $\pi A^2 l q^{(1)}$ be the liquid volume of the bounded by two flow planes separated by distance l. Ignoring the liquid-density variations due to the difference of T from T_s, we have

$$q^{(1)} = 1. \tag{29}$$

From (10), (11), (28), and (29) it follows that

$$a_0 = 4\sum_{n=1}^{\infty} b_{2n}\cos 2n\theta^*. \tag{30}$$

According to (28) and (30), we have

$$\xi_1^{(1)} = 0, \qquad \xi_2^{(1)} = 0. \tag{31}$$

Relations (21)–(25) and (31) define a solution of problem (10)–(16) that satisfies equality (29).

We note that for a rather smooth dependence of $v_\theta^{(1)}\big|_{r=1}$ on θ, the operations performed on the series are easily substantiated (see [5]).

Using (18)–(20), we obtain

$$\tau^{(1)} = c_0 + \sum_{n=1}^{\infty} c_n \cos 2n\theta\, r^{2n}, \tag{32}$$

where

$$c_0 = \int\limits_{\theta^*}^{\pi/2} \varphi d\theta; \quad c_n = \frac{4}{\pi} \int\limits_{\theta^*}^{\pi/2} \varphi \cos 2n\theta d\theta \qquad (n = 1, 2, \ldots).$$

The relations $\xi_1 = 1 + \mathrm{Ma}\, \xi_1^{(1)}$, $\xi_2 = 1 + \mathrm{Ma}\, \xi_2^{(1)}$, $\mathrm{v} = \mathrm{Ma}\, \mathrm{v}^{(1)}$, $p = \mathrm{Ma}\, p^{(1)}$, and $\tau = \mathrm{Ma}\, \tau^{(1)}$ and (21)–(25), (27), (31), and (32) define the approximate solution of the examined problem of the thermocapillary convection of a liquid in a floating zone.

The results obtained demonstrate the basic laws and provide answers to particular questions pertaining to the problem considered.

Other problems of plane or spatial convection of a liquid in a floating zone can be studied in the same manner as was done above. The approach proposed in the present paper can be used to study both steady and unsteady liquid flows in the absence or presence of mass forces.

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